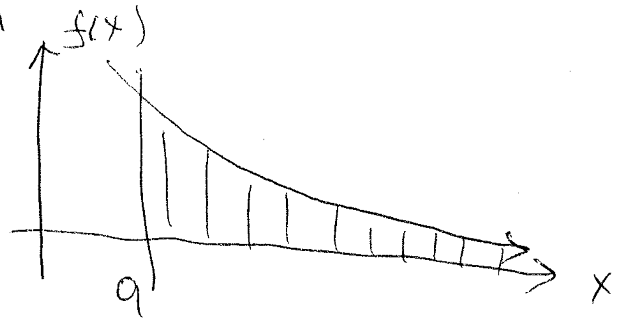
We looked at definite integrals $\int_a^b f(x)\,dx$ and we assumed that the interval $[a, b]$ was finite length and that f was continuous.

Very often integrals don't have these nice properties. There are two kinds of things that can happen, both related to infinity

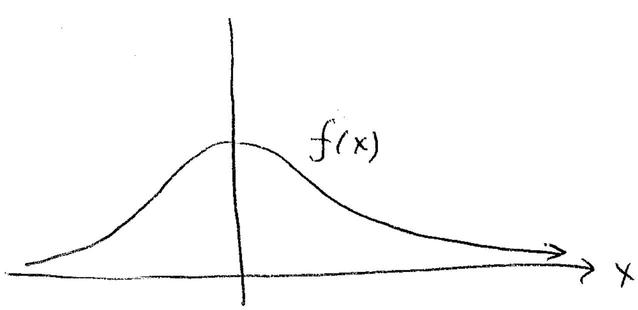
type I improper integrals

one (or both) limit(s) of integration is (are) infinite

$$\int_a^\infty f(x)\,dx$$



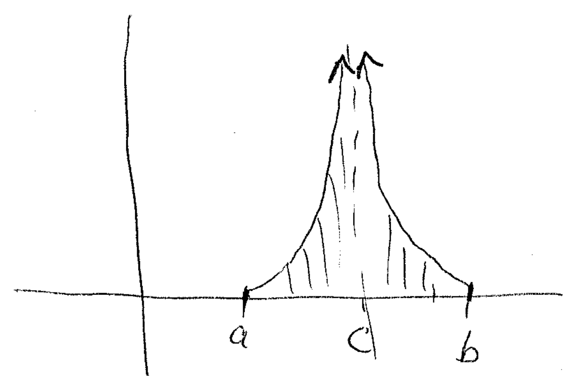
or



type II improper integr.

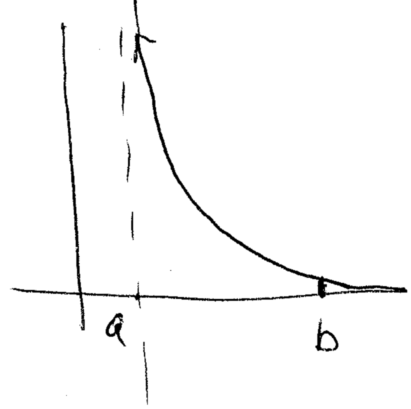
the integrand is unbounded

$$\int_a^b f(x)\,dx, \quad f(x) \to \infty$$
as $x \to c$, $c \in [a, b]$



(the function goes to ∞)

or



Type 1 Infinite limits of integration:

Definition

Let $f(x)$ be continuous on the interval $[a, \infty)$

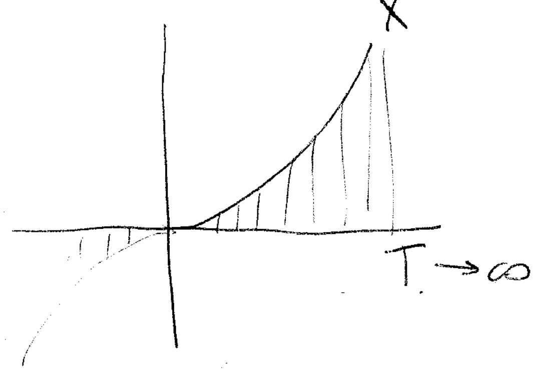
If $\lim_{T\to\infty} \int_a^T f(x)dx$ exists and has a $\boxed{\text{finite}}$ $\boxed{\text{value}}$, we say that the improper integral

$\int_a^\infty f(x)dx$ converges and define

$$\int_a^\infty f(x)dx = \lim_{T\to\infty} \int_a^T f(x)dx$$

Otherwise, we say that the improper integral diverges. (The integral is convergent (or divergent))

$$\int_0^\infty x^3 dx = \lim_{T\to\infty} \int_0^T x^3 dx = \lim_{T\to\infty} \left.\frac{x^4}{4}\right|_{x=0}^{x=T} = \lim_{T\to\infty} \frac{T^4}{4} \to \infty$$



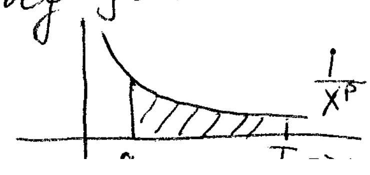
the function increases, $T \to \infty$.

The area of the region goes to infinity as $T \to \infty$

$\int_0^\infty x^3 dx$ is divergent.

- We will focus on decreasing functions of the form

$\underline{f(x) = \frac{1}{x^p}}$, $0 < P$,



, $\underline{g(x) = e^{-\alpha x}}$ $(\alpha > 0)$

The General Case (the First One)

Determine for which values of the exponent p, the improper integral $\int_1^{\infty} \frac{dx}{x^p}$ diverges.

$$\lim_{T\to\infty} \int_1^T \frac{dx}{x^p} = \lim_{T\to\infty} \int_1^T x^{-p}\,dx =$$

$$\lim_{T\to\infty} \left(\frac{x^{-p+1}}{-p+1} \Bigg|_{x=1}^{x=T} \right) = \lim_{T\to\infty} \frac{x^{1-p}}{1-p} \Bigg|_{x=1}^{x=T} =$$

$$= \frac{T^{1-p}}{1-p} - \frac{(1)^{1-p}}{1-p} = \frac{T^{1-p}}{1-p} - \frac{1}{1-p}.$$

$$T \to \infty$$

- If $1-p>0$. then T will be in the numerator and the whole expression will go to ∞. Thus, $1-p>0$ or $\boxed{p<1}$ $\Rightarrow$ the integral is divergent.
- If $1-p<0$ or $\boxed{p>1}$ $\Rightarrow$ the integral is convergent.
- If $p=1$ $\int_1^{\infty} \frac{dx}{x} = \lim_{T\to\infty} \int_1^T \frac{dx}{x} = \lim_{T\to\infty} \left(\ln|x| \Bigg|_{x=1}^{x=T} \right) =$

$$= (\ln T - \ln 1) \Rightarrow \infty \quad \Rightarrow \text{the integral is divergent as well.}$$

$$T\to\infty$$

To summarize:

$$\int_a^\infty \frac{dx}{x^p} \text{ is convergent when } p>1$$

and

$$\int_a^\infty \frac{dx}{x^p} \text{ is divergent when } p \leq 1$$

a is any finite number

For example, Does the integral converge or diverge

$$\int_1^\infty \frac{dx}{\sqrt[5]{x^{12}}} = \int_1^\infty \frac{dx}{x^{\frac{12}{5}}} = \lim_{T\to\infty} \int_1^T \frac{dx}{x^{\frac{12}{5}}} = \lim_{T\to\infty} \int_1^T x^{-\frac{12}{5}}\,dx =$$

$$= \lim_{T\to\infty} \left(\frac{x^{-\frac{12}{5}+1}}{-\frac{12}{5}+1} \Bigg|_{x=1}^{x=T} \right) = \lim_{T\to\infty} \left(\frac{x^{-\frac{7}{5}}}{-\frac{7}{5}} \right) \Bigg|_1^T =$$

$$= \lim_{T\to\infty} -\frac{5}{7x^{7/5}} \Bigg|_1^T = \lim_{T\to\infty} \left(-\frac{5}{7T^{7/5}} + \frac{5}{7(1)^{7/5}} \right) =$$

$$= \frac{5}{7}$$ ← the value of the improper integral

Remark: We have to accept the fact that some unbounded regions might have finite area

The Second General Case | Type 1.

$$g(x) = e^{-\alpha x}, \quad \alpha > 0$$

$$\int_0^\infty e^{-\alpha x}\,dx = \lim_{T\to\infty} \int_0^T e^{-\alpha x}\,dx =$$

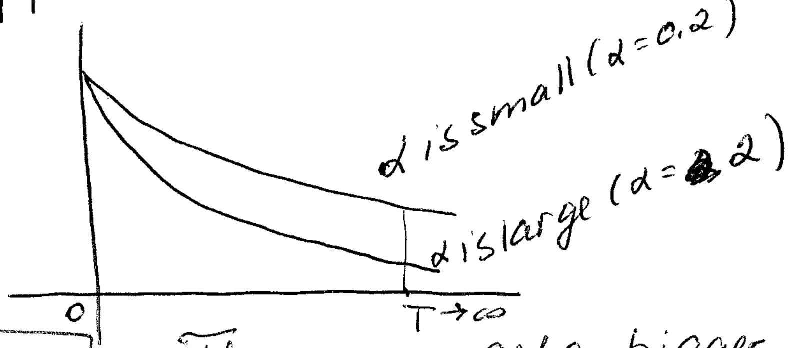
$$= \lim_{T\to\infty} \left(\frac{e^{-\alpha x}}{-\alpha} \Bigg|_{x=0}^{x=T} \right) =$$

$$= \lim_{T\to\infty} \left(-\frac{e^{-\alpha T}}{\alpha} + \frac{e^{-\alpha \cdot 0}}{\alpha} \right) =$$

$$= \lim_{T\to\infty} \left(-\frac{1}{\alpha e^{\alpha T}} + \frac{1}{\alpha} \right) = \frac{1}{\alpha}.$$

This integral converges for every positive value of α.

($e^{+\alpha T}$ grows very rapidly, so the area approaches $\frac{1}{\alpha}$ instead of growing without bound)



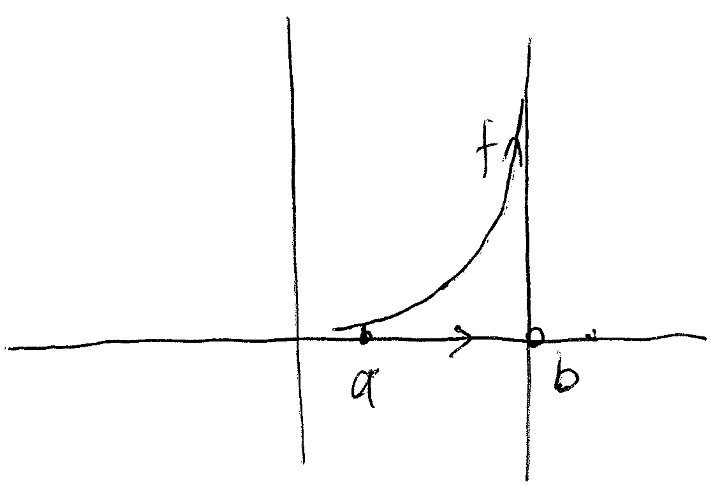
$\boxed{\text{Remark}}$ The area gets bigger as α gets smaller

$$\int_0^\infty e^{-2t}\,dt = \frac{1}{2} < \int_0^\infty e^{-0.2t}\,dt = 5$$

$(\alpha = 2) \qquad (\alpha = 0.2)$

There is another way for an integral to be improper. Namely, the interval may be finite but the function may be unbounded near some points in the interval.

Type 2, Infinite Integrands

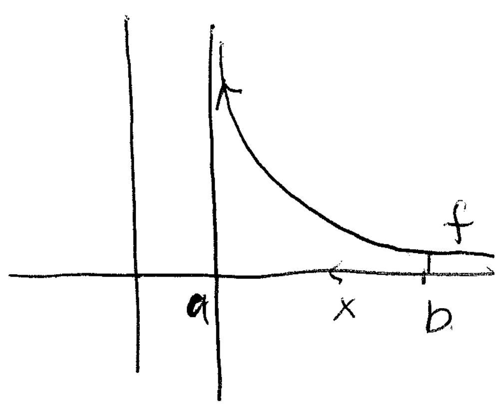


function f is given on $[a, b)$

$x \to b^-$ (x approache b from the left.)

$f(x) \to \infty$.

f is given on $(a, b]$.



$x \to a^+$ (x approaches a from the right)

$f(x) \to \infty$

$[a, b]$ is a closed interval (a, b are included)

(a, b) is an open interval (a, b are excluded)

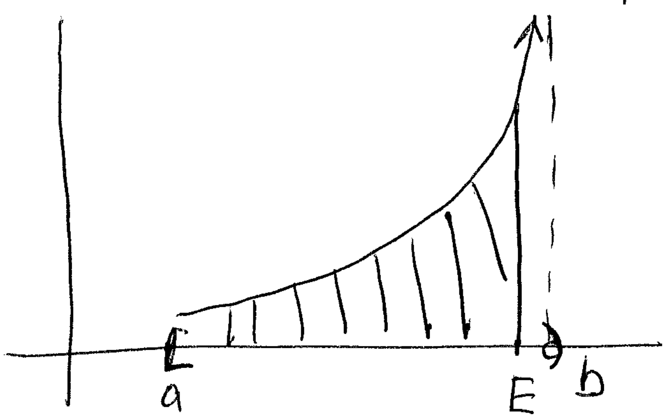
Definition Assume that $f(x)$ is continuous on $[a,b)$ but it is not continuous at $x=b$ (it tends to infinity as $x \to b^-$)

If $\lim\limits_{E \to b^-} \int_a^E f(x)\,dx$ is a <u>finite number</u>,

we say that $\int_a^b f(x)\,dx$ converges and

define $\int_a^b f(x)\,dx = \lim\limits_{E \to b^-} \int_a^E f(x)\,dx.$

Otherwise, we say $\int_a^b f(x)\,dx$ diverges.



$[a \longrightarrow b)$

, $E \to b^-$

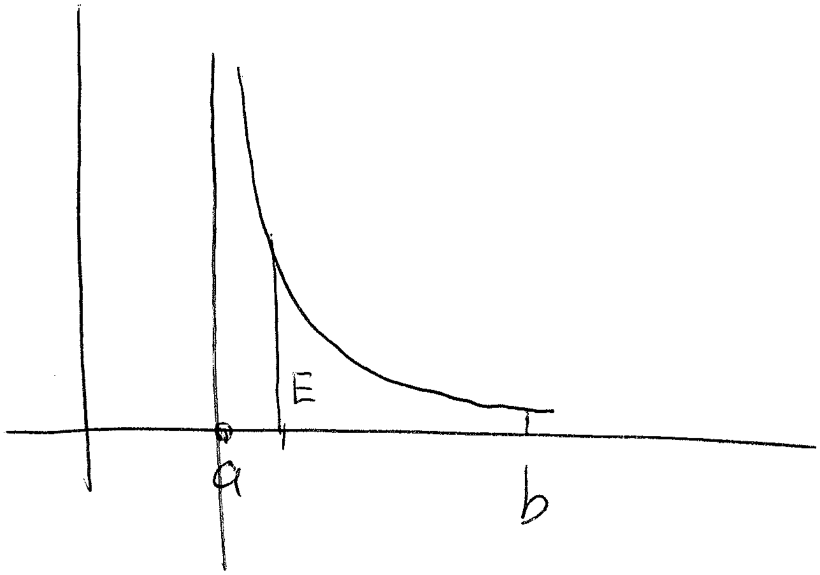
→ Similarly, if $f(x)$ is continuous on $(a,b]$, but not continuous at $x=a$

If $\lim\limits_{E \to a^+} \int_E^b f(x)\,dx$ is a finite number,

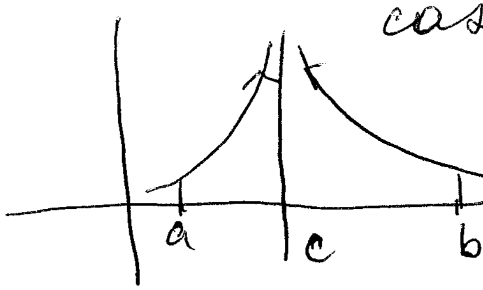
we say that $\int_a^b f(x)\,dx$ converges and

define $\int_a^b f(x)\,dx = \lim\limits_{E \to a^+} \int_E^b f(x)\,dx$,

Otherwise, we say $\int_a^b f(x)\,dx$ diverges



The following case is the combination of the above cases.



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

if <u>both</u> improper integral converge then the integral $\int_a^b f(x)\,dx$ converges.

Otherwise, it is divergent.

Example — The General Case:

$$f(x) = \frac{1}{x^p}, \quad p>0, \; p \neq 1. \quad \boxed{a=0.}$$

$$\int_0^1 \frac{dx}{x^p} = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \frac{dx}{x^p} = \lim_{\varepsilon \to 0^+} \left[\frac{x^{-p+1}}{-p+1} \Bigg|_{x=\varepsilon}^{x=1} \right] =$$

$$= \lim_{\varepsilon \to 0^+} \left[\frac{1}{1-p} - \frac{\varepsilon^{-p+1}}{1-p} \right].$$

$-p+1>0.$

$\boxed{p<1}$ $\Rightarrow$ the integral converges.

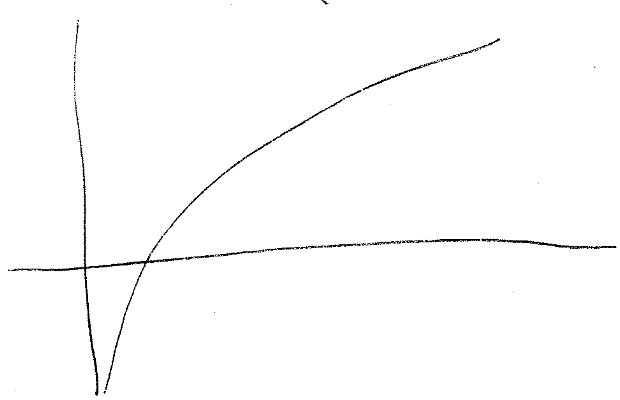
Otherwise, if $-p+1<0.$
$p>1. \Rightarrow$ it is divergent

- If $p=1$

$$\int_0^1 \frac{dx}{x} = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \frac{dx}{x} =$$

$$= \lim_{\varepsilon \to 0^+} \left[\ln x \Bigg|_{x=\varepsilon}^{x=1} \right] = \lim_{\varepsilon \to 0^+} \left[\ln 1 - \ln \varepsilon \right] =$$

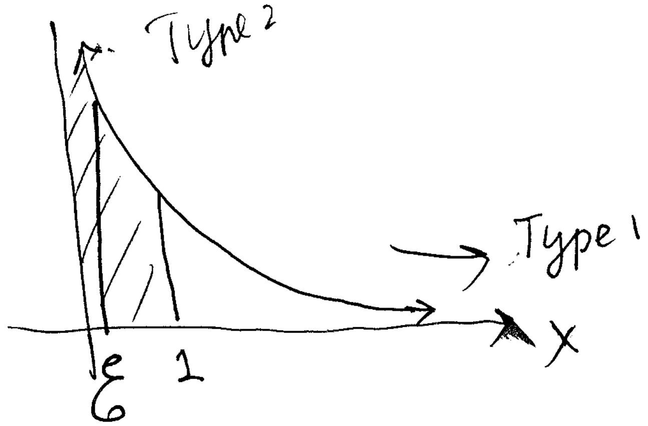
$$-(-\infty) = \infty.$$



$\Rightarrow p=1$ the integral is divergent as well.

$\boxed{\text{Example}}$ $f(x) = \frac{1}{\sqrt{x}}$ has a vertical asymptote at $x=0$. The region b/n the graph, the x-axis and the lines $x=0$ and $x = a$ (1) is unbounded.

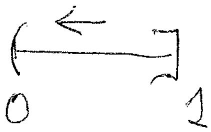
$\int_0^1 \frac{dx}{\sqrt{x}}$



Instead of extending to infinity in the horizontal direction as in the previous case (Type1), this region extends to infinity in the vertical direction.

Idea: We compute $\int_{\varepsilon}^{1} \frac{dx}{\sqrt{x}}$ for values of ε slightly larger than 0 and look at what happens as ε approaches 0 from the right ($\varepsilon \to 0^+$) ←

Example



$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 x^{-\frac{1}{2}}\,dx =$$

$$= \lim_{\varepsilon \to 0^+} \left[\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \Bigg|_\varepsilon^1 \right] = \lim_{\varepsilon \to 0^+} \left[2\sqrt{x} \Big|_\varepsilon^1 \right] =$$

$$= \lim_{\varepsilon \to 0^+} \left[2\sqrt{1} - 2\sqrt{\varepsilon} \right] = 2.$$

The integral converges.

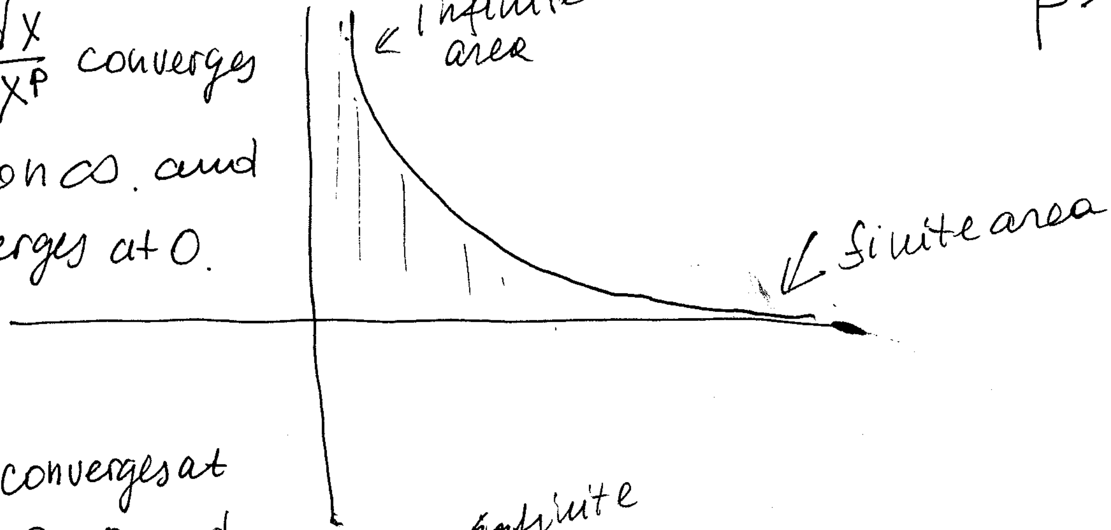
Example: $g(x) = \frac{1}{x^2}$.

$$\int_0^1 \frac{dx}{x^2} = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 x^{-2}\,dx = \lim_{\varepsilon \to 0^+} -\frac{1}{x} \Big|_{x=\varepsilon}^{x=1}$$

$$= \lim_{\varepsilon \to 0^+} \left(-1 + \frac{1}{\varepsilon} \right) = \infty.$$

— An infinite Area under a function that approaches ∞.

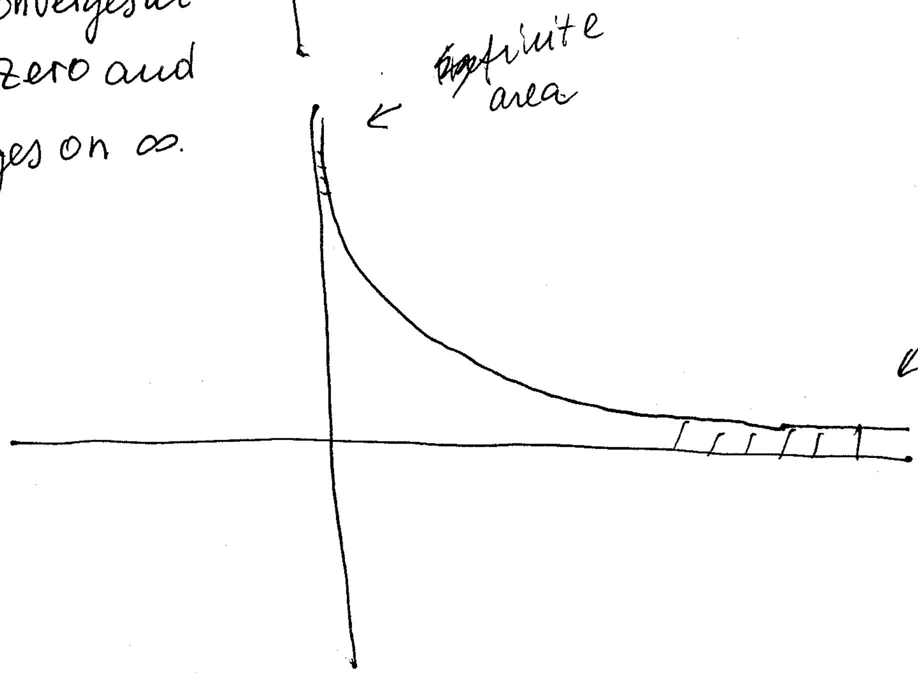
$\int_0^{\infty} \frac{dx}{x^p}$ converges on ∞ and diverges at 0.



$p>1$

$\int_0^{\infty} \frac{dx}{x^p}$ diverges. $p>1$

$\int_0^{\infty} \frac{dx}{x^p}$ converges at zero and diverges on ∞.



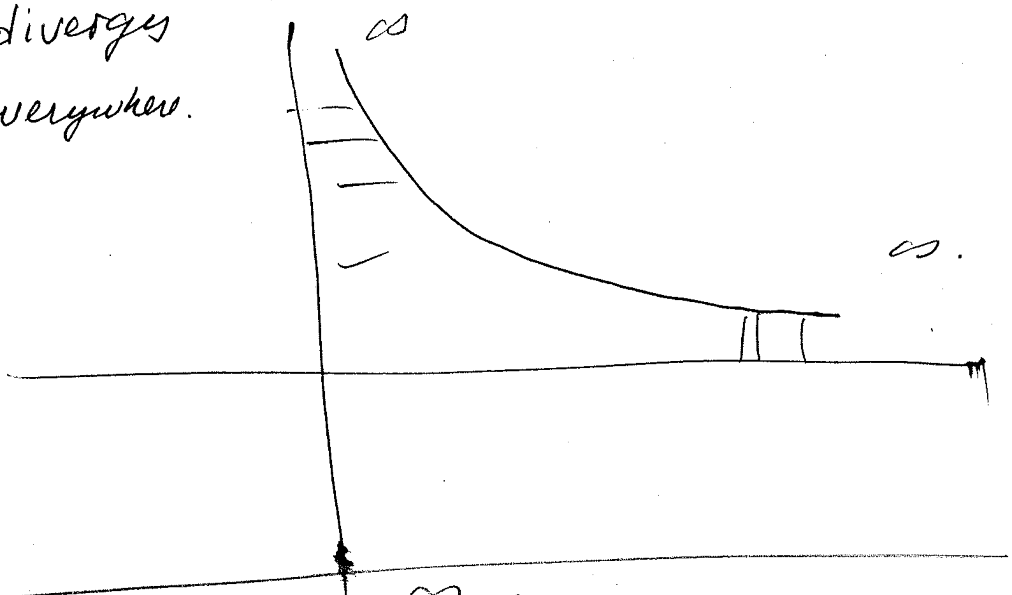
$p<1$

$\int_0^{\infty} \frac{dx}{x^p}$ $p<1$ diverges

$p=1$

$\int_0^{a} \frac{dx}{x}$ diverges everywhere.

$\int_a^{\infty} \frac{dx}{x}$



$$\int_0^{\infty} \frac{dx}{x^p} = \int_0^{a} \frac{dx}{x^p} + \int_a^{\infty} \frac{dx}{x^p}.$$

Look at $f(x) = \frac{1}{x^p}$ and $g(x) = \frac{1}{(x-a)^p}$,

where a is a positive number, $p > 0$, $p \neq 1$.

Both of the integrals $\int_0^b \frac{dx}{x^p}$ and

$\int_a^b \frac{dx}{(x-a)^p}$ converge or diverge simultaneously.

Why?

We already know that $\int_0^b \frac{dx}{x^p}$ converges if

$p < 1$ and diverges if $p \geq 1$.

Let's make a change of variable in the second integral:

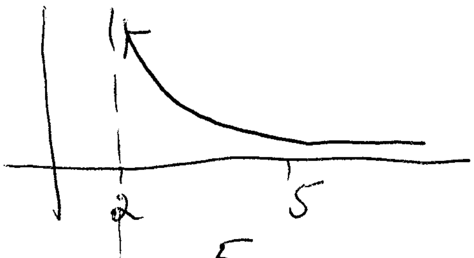
$$\boxed{\begin{array}{l} x - a = y \\ dx = dy \end{array} \quad \begin{array}{c|c|c} x & a & b \\ \hline y & 0 & b-a \end{array}}$$

$b-a$ is still a finite number

Then $\int_a^b \frac{dx}{(x-a)^p} = \int_0^{b-a} \frac{dy}{y^p}$ $\leftarrow$ this integral is exactly of the form $\int_0^b \frac{dx}{x^p}$.

Thus, $\int_a^b \frac{dx}{(x-a)^p}$ converges if $p < 1$ and diverges if $p \geq 1$.

Example $\displaystyle\int_2^5 \frac{dx}{\sqrt{x-2}} =$



$$= \lim_{E\to 2^+} \int_E^5 \frac{dx}{\sqrt{x-2}} = \lim_{E\to 2^+} \int_E^5 (x-2)^{-\frac{1}{2}}\,dx =$$

$$= \lim_{E\to 2^+} \left(\frac{(x-2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \Bigg|_{x=E}^{x=5} \right) = \lim_{E\to 2^+} \left(2(x-2)^{\frac{1}{2}} \Big|_E^5 \right) =$$

$$= \lim_{E\to 2^+} \left(2\cdot 3^{\frac{1}{2}} - 2(E-2)^{\frac{1}{2}} \right) = 2\cdot 3^{\frac{1}{2}} = \underline{\underline{2\sqrt{3}}}$$

the integral converges

or

$$\int_2^5 \frac{dx}{\sqrt{x-2}} = \boxed{\begin{array}{ll} \text{Substitution} & x-2 = u(x) \\ & dx = du \\ & \begin{array}{c|c|c} x & 2 & 5 \\ \hline u & 0 & 3 \end{array} \end{array}} =$$

$p = \frac{1}{2}$.

$$= \int_0^3 \frac{du}{u^{\frac{1}{2}}} = \lim_{\varepsilon\to 0^+} \int_\varepsilon^3 u^{-\frac{1}{2}}\,du = \lim_{\varepsilon\to 0^+} \left(\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right) \Bigg|_{u=\varepsilon}^{u=3} =$$

$$= \lim_{\varepsilon\to 0^+} \left(2u^{\frac{1}{2}}\right)\Big|_\varepsilon^3 = \underset{\varepsilon\to 0}{\left(2\cdot 3^{\frac{1}{2}} - 2\cdot\varepsilon^{\frac{1}{2}}\right)} = \underline{2\sqrt{3}}$$